RI-12-03
ITFA-2003-61

Comment on Counting Black Hole Microstates Using String Dualities

Assaf Shomer [1]

Racah Institute of Physics, The Hebrew University, Jerusalem 91904, Israel
and
Institute of Theoretical Physics, University of Amsterdam
Valckenierstraat 65, 1018 XE Amsterdam, The Netherlands[2].

We discuss a previous attempt at a microscopic counting of the entropy of asymptotically flat non-extremal black-holes. This method used string dualities to relate 4 and 5 dimensional black holes to the BTZ black hole. We show how the dualities can be justified in a certain limit, equivalent to a near horizon limit, but the resulting spacetime is no longer asymptotically flat.

December 2003

[1] E-mail : `shomer@science.uva.nl` ; `shomer@cc.huji.ac.il`.

[2] visiting.

1. Introduction.

One of the celebrated successes of string theory is the Strominger-Vafa [1] microscopic entropy counting for extremal 5 dimensional asymptotically flat black-holes (see [2] for review and references.). The same counting technique was successfully extended [3][4] also to some non-extremal black-holes, but the reason for the successful comparison in those cases is less clear due to open-string strong coupling effects. A different approach to this problem was presented in [5] where it was argued that 4 and 5 dimensional non-extremal black-holes can be related by symmetries of string theory to the 3 dimensional non-extremal BTZ black hole (for related works see [6]). The transformations did not change the horizon area so one could hope that, while indirect, this method reduces the counting problem to the BTZ case where we have a better control over the microstates (using e.g. AdS_3/CFT_2 [7][8] or Carlip's approach [9][3]).

However, as was already noticed [5] (see e.g. section 4.2.2 of [10] for a discussion) this line of argumentation suffers from a caveat. It crucially relies on a combination of T-duality and a coordinate transformation, referred to as the "shift" transformation, which effectively replaced the black-hole with its near horizon [11] (see also [12]). Closer inspection reveals that this involves a T-duality along a non-compact isometry. Moreover, the corresponding Killing vector was null at spatial infinity. This operation relates different solutions of the low energy supergravity but it is not entirely clear if such a transformation is a symmetry of string theory. Therefore, one could not argue for an equivalence between the initial and final configurations and the matching of the thermodynamic quantities was left as a suggestive indication that the two are somehow related. In this note we present a way of closing this loophole, so that all the duality transformations are well defined. This forces the introduction of a certain limiting procedure which appears different but turns out to be equivalent to the near horizon limit used by [7][13]. This explains the matching of the thermodynamic quantities but also shows that the counting is really done for a black-hole which is not asymptotically flat. We chose to focus on the 5 dimensional case, but the result is easily extendible to other cases.

The structure of this note is as follows. In section 2 we review the "shift" transformation in a simple setting, discuss its shortcomings and show how one can replace it by a limit. In section 3 we apply this to the non-extremal $D1-D5-$Wave in type IIB string

[3] This approach is somewhat controversial due to the inclusion of negative norm states in the entropy counting and the choice of boundary conditions on the horizon.

theory and get the BTZ$\times S^3 \times T^4$ background. In section 4 we discuss the nature of the limit and its relation to the near horizon limit. In section 5 we discuss the reduction to 5 dimensions. A short summary appears on section 6.

2. The Shift Transformation

In this section we quickly review the simplest case of the "Shift" transformation [11] for the fundamental string solution. We then argue that a possible way of extending it to a symmetry of string backgrounds is via a certain limiting procedure.

We start with the non-extremal black string solution of type II supergravity in the string frame[4]

$$\begin{aligned} ds^2 &= \frac{1}{H(r)}(-f(r)dt^2 + dx^2) + (\frac{1}{f(r)}dr^2 + r^2 d\Omega_7^2) \\ B_{tx} &= \frac{f(r)}{H(r)}\tanh\alpha \qquad ; \qquad e^{-2\phi} = \frac{H(r)}{g_s^2} \end{aligned} \tag{2.1}$$

where the harmonic function and the function controlling the non extremality are

$$H(r) = 1 + \frac{\mu^6}{r^6}\sinh^2\alpha \qquad ; \qquad f(r) = 1 - \frac{\mu^6}{r^6}. \tag{2.2}$$

This solution has an inner horizons at $r = 0$ and an outer horizon at $r = \mu$. The constant part of the antisymmetric tensor B is fixed so that it vanishes on the outer horizon as required by regularity [14]. The coordinate x is periodic with period R

$$x \sim x + 2\pi R. \tag{2.3}$$

The entropy is

$$\mathcal{S} = \frac{2\pi R \mu^7 \omega_7}{4G_N^{(10)}}\cosh\alpha \tag{2.4}$$

The idea behind the shift transformation is to go to the near horizon geometry (effectively to "drop the 1" from the harmonic function) using a chain of U-duality and coordinate transformations (assumed to be symmetry operations) in a way that leaves the entropy invariant.

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[4] Throughout this note g_s stands for the asymptotic value of the dilaton and ω_d for the volume of the d dimensional unit sphere. We work in conventions where $G_N^{(10)} = 8\pi^6 g_s^2(\alpha')^4$.

Starting with a T-duality along x we get

$$
\begin{aligned}
&ds^2 = -f(\frac{dt}{\cosh\alpha} - \sinh\alpha\ dx)^2 + \cosh^2\alpha\ dx^2 + \frac{1}{f}dr^2 + r^2 d\Omega_7^2 \\
&B = 0 \qquad ; \qquad e^{-2\phi} = \frac{R^2}{g_s^2}.
\end{aligned}
\tag{2.5}
$$

Now perform the following $SL(2,R)$ change of variables. One uses an $SL(2,R)$ transformation in order not to change the area of the horizon and thus the entropy of the solution.

$$
\tilde{t} \equiv \frac{1}{\cosh\alpha}t + e^{-\alpha}x \quad ; \quad \tilde{x} \equiv (\cosh\alpha)x. \tag{2.6}
$$

In terms of the new (tilded) variables the solution takes the form

$$
\begin{aligned}
&ds^2 = -f(d\tilde{t} - d\tilde{x})^2 + d\tilde{x}^2 + \frac{1}{f}dr^2 + r^2 d\Omega_7^2 \\
&B = 0 \qquad ; \qquad e^{-2\phi} = \frac{R^2}{g_s^2}.
\end{aligned}
\tag{2.7}
$$

T-dualizing back along $\tilde{x}$ gives:

$$
\begin{aligned}
&ds^2 = \frac{1}{1-f}(-f d\tilde{t}^2 + d\tilde{x}^2) + (\frac{1}{f}dr^2 + r^2 d\Omega_7^2) \\
&B_{\tilde{t}\tilde{x}} = \frac{f}{1-f} \qquad ; \qquad e^{-2\phi} = \frac{1-f}{g_s^2}.
\end{aligned}
\tag{2.8}
$$

Comparing (2.8) with (2.1) we see the chain of transformation amounts to replacing $H(r) \to 1 - f(r) = \frac{\mu^6}{r^6}$ namely dropping the 1 in the harmonic function in (2.2) and rescaling by $\sinh^2\alpha$. Since both T [14] and S duality transformations do not change the area of the horizon and since (2.6) also leaves it invariant one can argue that the above duality chain leaves the entropy invariant (*isentropic*) [5]. This technique was extended [5] to relate 4 and 5 dimensional black holes with the BTZ black hole where one can then use various techniques for microscopic entropy counting (see [10] for a review.)

However, this strategy, while very suggestive, has a loophole. After the coordinate transformation (2.6) the orbit of the killing vector $\partial_{\tilde{x}}$ is non-compact[5]. Instead (2.3) induces the following identification

[5] This is a little confusing because naively $\tilde{x}$ parameterizes a rescaled circle. However, if the orbits of $\partial_{\tilde{x}}$ were compact then, starting from a point, by just going along the orbit of $\partial_{\tilde{x}}$ keeping $\tilde{t}$ fixed we must reach another point, identified with the original under (2.3). But keeping $\tilde{t}$ fixed we move along the non-compact orbit of the coordinate t which never returns to the original point.

$$\tilde{x} \sim \tilde{x} + 2\pi \frac{\cosh\alpha}{R} \qquad \wedge \qquad \tilde{t} \sim \tilde{t} + 2\pi \frac{e^{-\alpha}}{R} \tag{2.9}$$

Therefore the T-duality transformation from (2.7) to (2.8) is not a symmetry. Moreover, the norm of the killing vector is $|\partial_{\tilde{x}}|^2 = \frac{\mu^6}{r^6}$ and so this isometry (while spacelike at any finite distance from the singularity) becomes null at spatial infinity[6]. T-duality with respect to isometries that are not everywhere spacelike is currently less well understood.

The main point of this comment is to suggest a strategy for closing this loophole. We can make $\tilde{x}$ compact while retaining a non-singular solution by taking the following limit

$$\begin{aligned} &R\ ,\ \alpha\ ,\ g_s \to \infty \\ &\mu\ ,\ r\ ,\ \frac{R}{\cosh\alpha} \equiv \tilde{R}\ ,\ \frac{g_s}{\cosh\alpha} \equiv \tilde{g}_s \quad \sim fixed. \end{aligned} \tag{2.10}$$

In this limit we get

$$\begin{aligned} &ds^2 = -f(d\tilde{t} - d\tilde{x})^2 + d\tilde{x}^2 + \frac{1}{f}dr^2 + r^2 d\Omega_7^2 \\ &B = 0 \qquad ; \qquad e^{-2\phi} = \frac{\tilde{R}^2}{\tilde{g}_s^2}, \end{aligned} \tag{2.11}$$

which is very similar to (2.7) but now $R \to \tilde{R}$, $g_s \to \tilde{g}_s$ and most importantly $\tilde{x} \sim \tilde{x} + 2\pi\tilde{R}$, namely, $\tilde{x}$ parameterizes a circle of radius $\tilde{R}$. Now we can safely T-dualize back along $\partial_{\tilde{x}}$ to get

$$\begin{aligned} &ds^2 = \frac{1}{1-f}(-fd\tilde{t}^2 + d\tilde{x}^2) + (\frac{1}{f}dr^2 + r^2 d\Omega_7^2) \\ &B_{\tilde{t}\tilde{x}} = \frac{f}{1-f} \qquad ; \qquad e^{-2\phi} = \frac{1-f}{\tilde{g}_s^2}. \end{aligned} \tag{2.12}$$

Comparing with (2.1) we see the effect of dropping the 1 from the harmonic function as before but now all the duality transformations are symmetries of string theory. Of course, there is a price to be paid because we took a limit and so the actual statement of duality will apply only to a limiting set of configurations.

We now notice that one can reach the endpoint (2.12) by directly taking the limit (2.10) in (2.1). When $\alpha \gg 1$ we can write $\Omega \equiv \sinh\alpha \sim \cosh\alpha \to \infty$ so $H(r) \sim (1-f)\Omega^2$ and asymptotically (2.1) becomes

[6] This is similar to the case of T-dualizing with respect to the angular isometry of the plane in polar coordinate.

$$
\begin{aligned}
ds^2 =& \frac{1}{(1-f)\Omega^2}(-fdt^2 + R^2 d\theta^2) + \frac{1}{f}dr^2 + r^2 d\Omega_7^2 + \mathcal{O}(\frac{1}{\Omega^2}) \rightarrow \\
& \frac{1}{(1-f)}(-fd\tilde{t}^2 + \tilde{R}^2 d\theta^2) + \frac{1}{f}dr^2 + r^2 d\Omega_7^2
\end{aligned} \tag{2.13}
$$

where we rescaled the time coordinate

$$
\tilde{t} \equiv \frac{t}{\Omega} \tag{2.14}
$$

in a manner resembling (2.6). Note that now the angle θ always parameterizes a circle. The B field and the dilaton also agree in this limit with (2.8).

The entropy of (2.12) is

$$
\tilde{\mathcal{S}} = \frac{A_8}{4G_N^{(10)}} = \frac{\tilde{R}\mu^7\omega_7}{16\pi^5 \tilde{g}_s^2 (\alpha')^4} \tag{2.15}
$$

which exactly equals (2.4). At any finite stage during the limit the entropies agree only up to terms that vanish exponentially with α due to the subleading term in (2.13) but the limiting configuration (2.12) has the same entropy as (2.1).

To summarize, the duality chain connecting (2.1) and (2.8) can be corrected to include only symmetry operations if one takes a certain limit. We then "forget" about the duality chain itself and define a limiting procedure that brings us directly from (2.1) to (2.8). The limiting configuration is the near horizon limit of the original configuration up to several rescalings of parameters. Both configurations have the same entropy. The specific example discussed above was used to explain our procedure but in itself suffers from some problems. For instance one needs to change to a weakly coupled description after the limit. We will not discuss this system any further in this note but rather apply the lesson in the D1-D5 system.

3. The D1-D5 system and 5 dimensional Black Holes

We now extend the discussion of the last section to the analysis of [5] connecting 5 dimensional black holes with the BTZ black hole. Start from the non-extremal $D1-D5$ system where the world-volume of the $D1$ brane is wrapped on a circle of radius R and carries some units of momentum along this circle. The remaining 4 directions in the $D5$

world-volume are compactified on a torus with Radii R_1, R_2, R_3, R_4. We write the solution in the string frame[7]

$$
\begin{aligned}
\alpha' d\sigma^2 &= \frac{1}{\sqrt{H_1(r)H_5(r)}}\left(-\frac{f(r)}{K(r)}dt^2 + K(r)(Rd\theta + \frac{1-f(r)}{K(r)}\sinh\alpha_k\cosh\alpha_k dt)^2\right) \\
&+ \sqrt{\frac{H_1(r)}{H_5(r)}}(R_1^2 d\psi_1^2 + \ldots + R_4^2 d\psi_4^2) + \sqrt{H_1(r)H_5(r)}(\frac{1}{f(r)}dr^2 + r^2 d\Omega_3^2) \\
\mathcal{C}^{(2)}_{t\theta} &= \frac{1}{g_s}\frac{R}{\alpha'}\frac{f(r)}{H_1(r)}\tanh\alpha_1 \qquad ; \quad e^{-2\phi} = \frac{H_5(r)}{g_s^2 H_1(r)} \\
\mathcal{C}^{(6)}_{t\theta\psi_1\ldots\psi_4} &= \frac{v}{g_s}\frac{R}{\alpha'}\frac{f(r)}{H_5(r)}\tanh\alpha_5.
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
H_{1/5}(r) &= 1 + \frac{\mu^2}{r^2}\sinh^2\alpha_{1/5} \ ; \ K(r) = 1 + \frac{\mu^2}{r^2}\sinh^2\alpha_k \\
f(r) &= 1 - \frac{\mu^2}{r^2} \ ; \ v = \frac{V}{(\alpha')^2} = \frac{R_1R_2R_3R_4}{(\alpha')^2}
\end{aligned}
\tag{3.2}
$$

and the rest of the IIB supergravity fields vanish.

This configuration has 3 conserved charges

$$
\begin{aligned}
\mathcal{N}_1 &= \frac{v\mu^2}{g_s\alpha'}\sinh\alpha_1\cosh\alpha_1, \\
\mathcal{N}_5 &= \frac{\mu^2}{g_s\alpha'}\sinh\alpha_5\cosh\alpha_5, \\
\mathcal{N}_k &= \frac{R^2 v\mu^2}{g_s^2(\alpha')^2}\sinh\alpha_k\cosh\alpha_k,
\end{aligned}
\tag{3.3}
$$

and the ADM mass, entropy and temperature are given by

$$
\begin{aligned}
\mathcal{M} &= \frac{Rv\mu^2}{2g_s^2(\alpha')^2}(\cosh 2\alpha_1 + \cosh 2\alpha_5 + \cosh 2\alpha_k), \\
\mathcal{S} &= \frac{2\pi Rv\mu^3}{g_s^2(\alpha')^2}\cosh\alpha_1\cosh\alpha_5\cosh\alpha_k, \\
\mathcal{T} &= \frac{1}{2\pi\mu\cosh\alpha_1\cosh\alpha_5\cosh\alpha_k}.
\end{aligned}
\tag{3.4}
$$

This configuration can be seen as a thermal excitation of the supersymmetric $D1-D5$ configuration and therefore satisfies a BPS condition

[7] We denote throughout the dimensionless metric by $d\sigma$ and the dimensionful metric by ds.

$$\mathcal{M} \geq \frac{\mathcal{N}_k}{R} + \frac{R}{g_s \alpha'}(\mathcal{N}_1 + v\mathcal{N}_5) \tag{3.5}$$

satisfied here thanks to the property $\cosh\alpha \geq \sinh\alpha$ (equality holding only for the asymptotic value $\alpha = \infty$).

In [5] a duality chain (including T-duality along non compact orbits) was used to relate this solution to the BTZ black-hole [15]. As in the previous section, we can close the loophole in the dualities for the price of taking a limit which in turn we can think of as an independent and equivalent way of performing the dualities. The limit in this case is deduced in a similar way to (2.10):

$$\begin{aligned} &\Omega_{1,5} \equiv \cosh\alpha_{1,5} \sim \sinh\alpha_{1,5} \to \infty \\ &\alpha' \ , R \ , R_{1,2,3,4} \to \infty \end{aligned} \tag{3.6}$$

keeping the following quantities fixed

$$\tilde{\alpha'} \equiv \frac{\alpha'}{\Omega_1\Omega_5} \ , \ \tilde{R} \equiv \frac{R}{\Omega_1\Omega_5} \ , \ \tilde{R_i} \equiv \frac{R_i}{\Omega_5} \ , \ \tilde{g_s} \equiv g_s\frac{\Omega_1}{\Omega_5} \tag{3.7}$$

As in (2.14) we have to rescale the time coordinate

$$\tilde{t} \equiv \frac{t}{\Omega_1\Omega_5} \Rightarrow \tilde{E} \equiv \Omega_1\Omega_5 E. \tag{3.8}$$

As opposed to (2.10) we do not need to send $g_s \to \infty$ since the dilaton in (3.1) involves a ratio of the harmonic functions but we do need to send $\alpha' \to \infty$ due to an overall $\Omega_1\Omega_5$ factor in front of the metric. The common feature in both cases is a rescaling of the Planck mass. Note also that even though $l_p \to \infty$ this is a low energy limit[8] because from (3.8) we see that $\frac{E}{m_p} \sim \frac{1}{\sqrt{\Omega_1\Omega_5}} \to 0$.

Taking the limit while keeping all tilded quantities finite we get the following solution

$$\begin{aligned} \tilde{\alpha}' d\sigma^2 &= \frac{1}{1-f}\left(-\frac{f}{K}d\tilde{t}^2 + K(\tilde{R}d\theta + \frac{1-f}{K}\sinh\alpha_k\cosh\alpha_k d\tilde{t})^2\right) \\ &\quad + \tilde{R}_1^2 d\psi_1^2 + \ldots + \tilde{R}_4^2 d\psi_4^2 + (1-f)(\frac{1}{f}dr^2 + r^2 d\Omega_3^2) \\ \mathcal{C}^{(2)}_{\tilde{t}\theta} &= \frac{1}{\tilde{g}_s}\frac{\tilde{R}}{\tilde{\alpha}'}\frac{f}{1-f} \qquad ; \quad e^{-2\phi} = \frac{1}{\tilde{g}_s^2} \\ \mathcal{C}^{(6)}_{\tilde{t}\theta\psi_1\ldots\psi_4} &= \frac{\tilde{v}}{\tilde{g}_s}\frac{\tilde{R}}{\tilde{\alpha}'}\frac{f}{1-f}. \end{aligned} \tag{3.9}$$

[8] This is like in M(atrix) theory where $\alpha' \to \infty$ before scaling the Planck mass due to the vanishing spacelike circle [16].

Writing the r dependence explicitly in the Einstein frame we get

$$\sqrt{\tilde{g}_s}\tilde{\alpha}' d\sigma_E^2 = -\frac{\frac{r^2}{\mu^2}-1}{1+\frac{\mu^2}{r^2}\sinh^2\alpha_k}d\tilde{t}^2 + (\frac{r^2}{\mu^2}+\sinh^2\alpha_k)\left(\tilde{R}d\theta + \frac{\mu^2\sinh\alpha_k\cosh\alpha_k}{(r^2+\mu^2\sinh^2\alpha_k)}d\tilde{t}\right)^2$$
$$+\frac{\mu^2}{r^2-\mu^2}dr^2 + \mu^2 d\Omega_3^2 + \tilde{R}_1^2 d\psi_1^2 \ldots + \tilde{R}_4^2 d\psi_4^2, \tag{3.10}$$

the $\tilde{t}, \theta, r$ part of the solution is identified with the general non-extremal BTZ black-hole solution of 3d gravity with negative cosmological constant $\Lambda = -\frac{1}{l^2}$ [15]

$$ds_{BTZ}^2 = -\frac{(\rho^2-\rho_+^2)(\rho^2-\rho_-^2)}{l^2\rho^2}d\tau^2 + \rho^2(d\varphi^2 + \frac{\rho_+\rho_-}{l\rho^2}d\tau)^2 + \frac{l^2\rho^2}{(\rho^2-\rho_+^2)(\rho^2-\rho_-^2)}d\rho^2 \tag{3.11}$$

where we identified the Planck scales and used the following dictionary

$$\begin{gathered} l_{BTZ} = \mu \quad ; \quad \tau_{BTZ} = \tilde{t}\frac{\mu}{\tilde{R}} \quad ; \quad \rho_{BTZ}^2 = \frac{\tilde{R}^2}{\mu^2}(r^2+\mu^2\sinh^2\alpha_k) \\ \rho_+ = \tilde{R}\cosh\alpha_k \qquad ; \qquad \rho_- = \tilde{R}\sinh\alpha_k \end{gathered} \tag{3.12}$$

The full solution (3.9)(3.10) is thus BTZ$\times S^3 \times T^4$.

4. The limit

The charges and the thermodynamical quantities associated with (3.10)(3.9) can be inferred from the BTZ factor to be[9]

$$\begin{aligned}
\tilde{\mathcal{M}} &= \frac{\mu}{\tilde{R}}\frac{\rho_+^2+\rho_-^2}{8l^2G_N^{(3)}} = \frac{\mu}{\tilde{R}}\frac{\tilde{R}^2(\cosh^2\alpha_k+\sinh^2\alpha_k)(2\pi)^4\tilde{v}(2\pi^2)\mu^3}{8\mu^2(8\pi^6\tilde{g_s}^2(\tilde{\alpha}')^2)} = \frac{\tilde{R}\tilde{v}\mu^2}{2\tilde{g_s}^2(\tilde{\alpha}')^2}\cosh 2\alpha_k \\
\tilde{\mathcal{J}} &= \frac{2\rho_+\rho_-}{8lG_N^{(3)}} = \frac{\tilde{R}^2\tilde{v}\mu^2}{\tilde{g_s}^2(\tilde{\alpha}')^2}\sinh\alpha_k\cosh\alpha_k \\
\tilde{\mathcal{S}} &= \frac{2\pi\tilde{R}\tilde{v}\mu^3}{\tilde{g}_s^2(\tilde{\alpha}')^2}\cosh\alpha_k \\
\tilde{\mathcal{T}} &= \frac{\mu}{\tilde{R}}\frac{\rho_+^2-\rho_-^2}{2\pi\rho_+l^2} = \frac{1}{2\pi\mu\cosh\alpha_k}
\end{aligned} \tag{4.1}$$

[9] Note that dimensionful quantities had to be rescaled by powers of $\frac{\mu}{\tilde{R}}$ due to the transformation $\tilde{E} = \frac{\mu}{\tilde{R}}E_{BTZ}$ induced from (3.12).

The entropy and temperature in (4.1) are exactly the limiting values of those in (3.4)[10]. The charges in (3.3) are also finite in the limit with $\mathcal{N}_k$ approaching the BTZ angular momentum charge $\tilde{\mathcal{J}}$. Only the ratio of the two remaining charges encodes independent information about (3.9), namely, the dimensionless volume[11] of the T^4.

In summary

$$\begin{aligned} &\mathcal{N}_1 \to \frac{\tilde{v}\mu^2}{\tilde{g}_s\tilde{\alpha}'} \ ; \ \mathcal{N}_5 \to \frac{\mu^2}{\tilde{g}_s\tilde{\alpha}'} \ ; \ \frac{\mathcal{N}_1}{\mathcal{N}_5} = \tilde{v}, \\ &\mathcal{N}_k \to \tilde{\mathcal{J}} \ ; \ \mathcal{S} \to \tilde{\mathcal{S}} \ ; \ \mathcal{T} \to \tilde{\mathcal{T}}. \end{aligned} \tag{4.2}$$

The mass, however, diverges after we rescale the energy. This is not a problem since also the BPS "zero point" energy corresponding to the supersymmetric solution (3.5) diverges in exactly the same way. The physical information resides in the fluctuations above the BPS mass which when appropriately rescaled as dictated by (3.6)(3.7)(3.8) remain finite[12]

$$\Omega_1\Omega_5 \left(\mathcal{M} - \frac{R}{g_s\alpha'}(\mathcal{N}_1 + v\mathcal{N}_5) \right) \longrightarrow \tilde{\mathcal{M}} \geq \frac{\tilde{\mathcal{J}}}{\tilde{R}}. \tag{4.3}$$

The last inequality is the BPS condition for BTZ black-holes which follows in the limit from (3.5).

Actually, although it appears different, this limit gives exactly the same result as the "Near Inner Horizon Limit" (NIHL) for non-extremal branes defined in [7][13]. This limit drops the 1 from H_1, H_5 by sending $\alpha' \to 0$ keeping fixed

$$U = \frac{r}{\alpha'} \ ; \ U_0 = \frac{\mu}{\alpha'} \ ; \ R \ ; \ v \ ; \ g_s, \tag{4.4}$$

which in turn keep the charges $\mathcal{N}_1$; $\mathcal{N}_5$; $\mathcal{N}_k$; $\mathcal{S}$; $\mathcal{T}$ fixed[13]. This is not surprising because, after all, the motivation behind the shift transformation was to replace (using dualities) the black-hole with its near horizon. It appears that to do this consistently one has to take the near horizon limit.

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[10] Note that one has to rescale units of energy as in (3.8).

[11] This dimensionless volume $\tilde{v}$ is related to that of the original T^4 in (3.1) through the ratio $\sqrt{\tilde{v}} = \sqrt{v}\frac{\Omega_1}{\Omega_5}$ which is an arbitrary constant in the limit.

[12] Again this is like the situation in M(atrix) theory where the mass of the $D0$ brane diverges but the light-cone Hamiltonian is scaled to remain finite [16] (see e.g. discussion in [17]).

[13] Notice that this "NIHL" necessarily involves the "dilute gas condition" $\alpha_1, \alpha_5 \to \infty$.

5. Reducing to 5 dimensions

When we reduce the asymptotically flat (3.1) on $\theta, \psi_1, \ldots, \psi_4$ we get a 5 dimensional non-extremal black hole. The metric is given in the Einstein frame by

$$\begin{aligned} ds_E^2 &= \lambda^{-\frac{2}{3}}(-f dt^2) + \lambda^{\frac{1}{3}}(\frac{1}{f}dr^2 + r^2 d\Omega_3^2) \\ \lambda(r) \equiv H_1 H_5 K &= (1 + \frac{\mu^2}{r^2}\sinh^2\alpha_1)(1 + \frac{\mu^2}{r^2}\sinh^2\alpha_5)(1 + \frac{\mu^2}{r^2}\sinh^2\alpha_k). \end{aligned} \tag{5.1}$$

Notice that the 5 dimensional Schwarzschield solution is also a member of this family with all charges equal to zero. The entropy of this solution is exactly equal to that of (3.1), namely the one in (3.4).

Taking the limit (3.6)(3.7)(3.8) in (5.1) or alternatively reducing (3.9) on $\theta, \psi_1, \ldots, \psi_4$ we get another 5 dimensional black hole with metric

$$\begin{aligned} ds_E^2 &= \tilde{\lambda}^{-\frac{2}{3}}(-f d\tilde{t}^2) + \tilde{\lambda}^{\frac{1}{3}}(\frac{1}{f}dr^2 + r^2 d\Omega_3^2) \\ \tilde{\lambda} &= (1-f)^2 K = (\frac{\mu}{r})^4(1 + \frac{\mu^2}{r^2}\sinh^2\alpha_k) \end{aligned} \tag{5.2}$$

whose entropy is equal to that of (3.9) in (4.1).

The condition for validity of the dimensional reduction are that one focuses on excitations that are well below the threshold for KK and winding modes. Since in the limit (3.6)(3.7)(3.8) we keep the tilded quantities fixed and send $\Omega_{1,5} \to \infty$ the only non-trivial condition is to have in the original $D1 - D5$ system (3.1) $RE \ll 1$[14].

6. Summary and Discussion

The main point in [5] was to use the shift transformation to relate (3.9) with (5.1). The thermodynamical quantities seemed to agree but since this transformation is not a symmetry one could not claim an equivalence. The path we have chosen here suggests instead to relate (3.9) and the limit of (5.1), namely (5.2). Focusing on the low energy excitations of the non-extremal $D1 - D5$ system (3.1) satisfying $RE \ll 1$ one can take a limit which gives the general non-extremal BTZ$\times S^3 \times T^4$ background (3.9) and which at the same time is well approximated by the dimensional reduction to the non-extremal

[14] This regime seems related to the one studied in [18][19].

(albeit not most general) 5 dimensional black hole (5.2). Unfortunately, although (5.2) is a limiting configuration in the family (5.1) of asymptotically flat black-holes, it is not asymptotically flat itself, but rather asymptotes to a space that is conformal to $AdS_2 \times S^3$. This is a reflection of the fact that (3.9) is the NHL of (3.1) and as such is not asymptotically flat but rather asymptotically AdS_3. The relation between (3.9) and (5.2) is not a duality but dimensional reduction along the angular isometry ∂_θ, just like (5.1) is a dimensional reduction of (3.1).

Nevertheless since the entropy of the configuration (5.1) is equal to that of (5.2) which is the result of the limit and since our limiting procedure itself is essentially a near-horizon limit it is plausible to argue that the entropy counting done in (5.2) or (3.9) using entropy counting techniques for the BTZ black hole is closely related to the counting of degrees of freedom in the original asymptotically flat configuration.

Acknowledgments

We like to thank Riccardo Argurio, Jan de Boer, Miranda Cheng, Emiliano Imeroni, Asad Naqvi and Jan Troost for discussions and comments on the manuscript. Special thanks to Kostas Skenderis for pointing out [11] as well as for many discussions and carefully reading the manuscript. Thanks also to Liat Maoz for a clear introduction to 3d gravity. This work is supported by a Clore fellowship.

References

[1] A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996) [hep-th/9601029].

[2] A. W. Peet, hep-th/0008241. J. M. Maldacena, hep-th/9607235.

[3] C. G. Callan and J. M. Maldacena, Nucl. Phys. B **472**, 591 (1996) [hep-th/9602043].

[4] G. T. Horowitz and A. Strominger, Phys. Rev. Lett. **77**, 2368 (1996) [arXiv:hep-th/9602051]. G. T. Horowitz, J. M. Maldacena and A. Strominger, Phys. Lett. B **383**, 151 (1996) [arXiv:hep-th/9603109].

[5] K. Sfetsos and K. Skenderis, Nucl. Phys. B **517**, 179 (1998) [hep-th/9711138].

[6] V. Balasubramanian and F. Larsen, Nucl. Phys. B **528**, 229 (1998) [arXiv:hep-th/9802198]. Y. Satoh, Phys. Rev. D **59**, 084010 (1999) [arXiv:hep-th/9810135]. M. Cvetic and F. Larsen, Nucl. Phys. B **531**, 239 (1998) [arXiv:hep-th/9805097].

[7] J. Maldacena, Adv. Theor. Math. Phys. **2** (1998) 231 ; Int. J. Theor. Phys. **38** (1998) 1113 ; hep-th/9711200.

[8] A. Strominger, JHEP **9802**, 009 (1998) [hep-th/9712251].

[9] S. Carlip, Phys. Rev. D **51**, 632 (1995) [gr-qc/9409052]. S. Carlip, Phys. Rev. D **55**, 878 (1997) [gr-qc/9606043].

[10] K. Skenderis, Lect. Notes Phys. **541**, 325 (2000) [hep-th/9901050].

[11] S. Hyun, J. Korean Phys. Soc. **33**, S532 (1998) [hep-th/9704005]. H. J. Boonstra, B. Peeters and K. Skenderis, Phys. Lett. B **411**, 59 (1997) [hep-th/9706192]. H. J. Boonstra, B. Peeters and K. Skenderis, Fortsch. Phys. **47**, 109 (1999) [hep-th/9801076].

[12] G. T. Horowitz and D. L. Welch, Phys. Rev. Lett. **71**, 328 (1993) [hep-th/9302126]. E. Alvarez, L. Alvarez-Gaume, J. L. F. Barbon and Y. Lozano, Nucl. Phys. B **415**, 71 (1994) [hep-th/9309039]. I. Bakas, Phys. Lett. B **343**, 103 (1995) [hep-th/9410104]. E. Bergshoeff and K. Behrndt, Class. Quant. Grav. **15**, 1801 (1998) [hep-th/9803090]. E. Cremmer, I. V. Lavrinenko, H. Lu, C. N. Pope, K. S. Stelle and T. A. Tran, Nucl. Phys. B **534**, 40 (1998) [hep-th/9803259].

[13] J. M. Maldacena and A. Strominger, JHEP **9812**, 005 (1998) [hep-th/9804085].

[14] G. T. Horowitz and D. L. Welch, Phys. Rev. D **49**, 590 (1994) [hep-th/9308077].

[15] M. Banados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992) [hep-th/9204099].

[16] N. Seiberg, Phys. Rev. Lett. **79**, 3577 (1997) [hep-th/9710009].

[17] W. Taylor, Rev. Mod. Phys. **73**, 419 (2001) [arXiv:hep-th/0101126]. A. Shomer, Phys. Rev. D **68**, 086002 (2003) [hep-th/0303055].

[18] A. Strominger, JHEP **9901**, 007 (1999) [hep-th/9809027].

[19] J. M. Maldacena, J. Michelson and A. Strominger, JHEP **9902**, 011 (1999) [arXiv:hep-th/9812073].